

# Emergence of Non-Axisymmetric Vortex in Strong-Coupling Chiral $p$ -Wave Superconductor

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**Abstract** We studied strong-coupling effect upon an isolated vortex in a two-dimensional chiral  $p$ -wave superconductor. We solved the Eilenberger equation for the quasiclassical Green's functions and the Éliashberg equation with single mode Einstein boson self-consistently. We calculated the free-energy of each obtained vortex, and found that a non-axisymmetric vortex metastably exists in some situation.

## 1 Introduction

The Migdal and Éliashberg theory of “strong-coupling superconductivity”[1,2,3] has been very successful in qualitative description of superconductivity in real materials[4,5,6]. For example, it explains the deviation from the universal value  $(2\Delta)/(k_B T_c) \approx 3.53$  in Bardeen-Cooper-Schrieffer (BCS) theory, or the dependence of critical magnetic field on the transition temperature. Moreover, in some situation, it is known that the strong-coupling effect is not only a quantitative but also a qualitative effect (e.g., Refs. [7,8]). The strong-coupling effect modifies the spectrum of the quasiparticles. Therefore, one can expect that it may change the structure of low-energy states within the vortices in type-II superconductors. As far as we know, however, there have been only a few studies of the strong-coupling effect on a vortex on the basis of microscopic theories.

Two-dimensional chiral  $p$ -wave superconductivity is considered to be realized in  $\text{Sr}_2\text{RuO}_4$  [9,10,11,12]. This state is topologically non-trivial and attracts much attention in these days. Within the vortices of this superconductor, reflected in the topology of this system, there is a zero-energy bound state, which is expected to be very robust against not-so-strong impurities[13,14,15,16,17,18,19,20,21]. Recently, the relationship between this robustness and the odd-frequency pairing also has been discussed[17,21,22].

In the present paper, we calculated the self-consistent Éliashberg equation to study how the strong-coupling feature affects the vortex of a chiral  $p$ -wave superconductor microscopically. We also calculated the free-energies of the vortices and discuss its stability.

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## 2 Methods

In this study, we consider an isolated vortex in the two-dimensional spinless chiral  $p$ -wave superconductor with isotropic Fermi surface. We use quasiclassical theory[23,24]; we assume that the product of the coherence length of the superconductor  $\xi$  and the Fermi wavevector  $\mathbf{k}_F$  is much larger than unity. The quasiclassical Green's function  $\check{g}$  is a  $2 \times 2$  matrix and obeys the Eilenberger equation

$$i\hbar\mathbf{v}_F \cdot \nabla \check{g}(i\varepsilon_n, \alpha, \mathbf{r}) + [i\hbar\varepsilon_n \check{\tau}_3 + (q/c)\mathbf{v}_F \cdot \mathbf{A} \check{\tau}_3 - \check{\Sigma}(i\varepsilon_n, \alpha, \mathbf{r}), \check{g}(i\varepsilon_n, \alpha, \mathbf{r})] = \check{0}, \quad (1)$$

where  $\varepsilon_n = (2n+1)\pi k_B T / \hbar$  are the Matsubara frequencies,  $\mathbf{v}_F$  is a Fermi velocity,  $\alpha$  denotes a direction of momentum on the Fermi surface such that  $\mathbf{k} = k_F(\cos \alpha, \sin \alpha)$ ,  $\check{\tau}_i$  ( $i = 0, \dots, 3$ ) are the Pauli matrices,  $q$  is the elementary charge,  $c$  is the speed of light,  $\mathbf{A}$  is the vector potential, and

$$\check{\Sigma} = \begin{pmatrix} \sigma & \Delta \\ -\Delta^* & -\sigma \end{pmatrix}, \quad (2)$$

is the self-energy. The quasiclassical Green's function satisfies the normalization condition  $\check{g}^2 = -\pi^2 \check{\tau}_0$  and its bulk value is

$$\check{g} = \frac{\pi}{\sqrt{-(i\hbar\varepsilon_n - \sigma)^2 + |\Delta|^2}} \begin{pmatrix} -i\hbar\varepsilon_n + \sigma & \Delta \\ -\Delta^* & i\hbar\varepsilon_n - \sigma \end{pmatrix}. \quad (3)$$

To incorporate strong-coupling effect, we use Éliashberg equation to calculate the self-energy  $\Sigma$  from the quasiclassical Green's function;

$$\check{\Sigma}(i\varepsilon_n, \alpha, \mathbf{r}) = N_0 k_B T \sum_{\varepsilon_m}^{| \varepsilon_m | < \varepsilon_c} \langle v(i\varepsilon_n, \alpha, i\varepsilon_m, \alpha') \check{g}(i\varepsilon_m, \alpha', \mathbf{r}) \rangle_{\alpha'}, \quad (4)$$

where  $\varepsilon_c$  is the cutoff of the Matsubara frequencies,  $N_0$  is the density of states on the Fermi level, and  $\langle \dots \rangle_\alpha$  denotes the average over the Fermi surface and is defined  $\langle A(\alpha) \rangle_\alpha = \int_0^{2\pi} d\alpha A(\alpha) / (2\pi)$ . We took 47 equally spaced points in the momentum space. We assume that the interaction between electrons  $v$  has the following form:

$$v(i\varepsilon_n, \alpha, i\varepsilon_m, \alpha') = \frac{C\omega_0^2}{(\varepsilon_n - \varepsilon_m)^2 + \omega_0^2} \times 2\cos(\alpha - \alpha'), \quad (5)$$

where  $\omega_0$  is a characteristic frequency of a mediated boson, and  $C$  is a constant parameter. We set these parameters so that  $\hbar\omega_0 = 3k_B T_c$ , where  $T_c$  is the critical temperature of the superconductivity. We choose this value so that the strong-coupling effect is very large but not unrealistic<sup>1</sup>. We set the cutoff of the Matsubara frequencies  $\hbar\varepsilon_c = 20k_B T_c$ , and confirm that this cutoff is considered to be sufficiently large by comparison of the magnitude of the bulk pair-potential with those for  $\hbar\varepsilon_c = 10k_B T_c$  and  $15k_B T_c$ . We also define  $\xi_0 = \hbar v_F / (k_B T_c)$  and use it as a characteristic length of the spatial modulation of the self-energy.

The vector potential  $\mathbf{A}$  is obtained from the quasiclassical Green's function as

$$\nabla \times (\nabla \times \mathbf{A}) = \frac{q}{c} N_0 k_B T \sum_{\varepsilon_n} \text{Tr} \langle \mathbf{v}_F \check{g}_{11} \rangle, \quad (6)$$

<sup>1</sup> With this parameter, the ratio of the energy gap to the critical temperature ( $2\Delta/T_c$ ) is about 5.6 at  $T = 0.02T_c$ . For example, CeCoIn<sub>5</sub> exhibits such a large value ( $\approx 6$ ) [25].

where  $\text{Tr}$  is a trace over the Nambu space. We define  $\lambda_0 = (N_0 v_F^2 q^2 c^{-2})^{-1/2}$  as a characteristic length of the electromagnetic entities. We set  $\lambda_0/\xi_0 = 2.5$  in this paper.

To discuss the stability of the isolated vortices, we calculated the free-energy deviation from the normal state  $\Omega_{\text{sn}}$  with the following equation:

$$\Omega_{\text{sn}} = \int d\mathbf{r} \left( N_0 k_B T \sum_{\epsilon_n} \text{Tr} \left\{ \int_0^1 ds \langle \check{g}_s \check{\Sigma} \rangle - \frac{1}{2} \langle \check{g} \check{\Sigma} \rangle \right\} + \frac{B^2}{2} \right), \quad (7)$$

where  $\mathbf{B} = \nabla \times \mathbf{A}$  is the magnetic field, and  $\check{g}_s$  is a solution of

$$i\hbar \mathbf{v}_F \cdot \nabla \check{g}_s + [i\hbar \epsilon_n \check{\tau}_3 + (q/c) \mathbf{v}_F \cdot \mathbf{A} \check{\tau}_3 - s \check{\Sigma}, \check{g}_s] = \check{0}, \quad \check{g}_s^2 = -\pi^2 \check{\tau}_0. \quad (8)$$

The above expression of  $\Omega_{\text{sn}}$  is a simple extension of the weak-coupling BCS one[26,27]. We used the 15-points Gauss-Kronrod quadrature formula to integrate respect to  $s$ .

We numerically confirmed that the self-energy for Matsubara-frequencies  $\check{\Sigma}(i\epsilon_n, \alpha, \mathbf{r})$  can be decoupled as

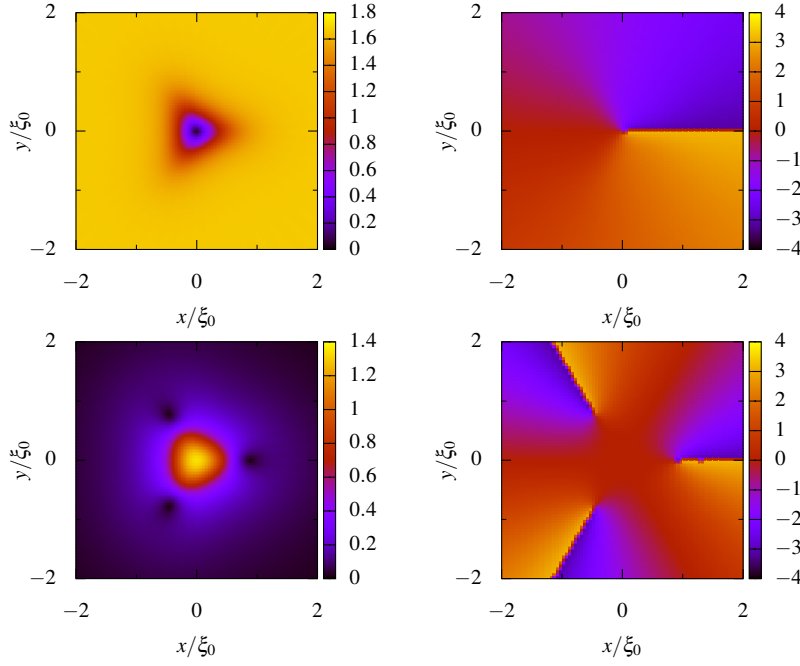
$$\check{\Sigma}(i\epsilon_n, \alpha, \mathbf{r}) = h(i\epsilon_n) \left[ \check{\Sigma}_+(\mathbf{r}) e^{+i\alpha} + \check{\Sigma}_-(\mathbf{r}) e^{-i\alpha} \right], \quad (9)$$

and thus we show only the  $\mathbf{r}$ -dependent part  $\check{\Sigma}_+(\mathbf{r})$  and  $\check{\Sigma}_-(\mathbf{r})$  in the following section. At sufficiently far from the vortex, only  $\check{\Sigma}_+$  or  $\check{\Sigma}_-$  survives. Hereafter we assume that  $\check{\Sigma}_+$  is a dominant part of the self-energy and survives in the bulk. As we can see in (9), Cooper pair of chiral  $p$ -wave superconductivity has internal angular momentum (chirality). If there is a vortex, two types of vortices can exist in this system; one type of vortex has vorticity (the angular momentum of vortex) parallel to the chirality, and the other type has vorticity anti-parallel to the chirality. In the present paper, we call the former “parallel vortex” and the latter “anti-parallel vortex”. In the Ginzburg-Landau(GL) theory, an anti-parallel vortex was shown to be more stable than a parallel vortex[28].

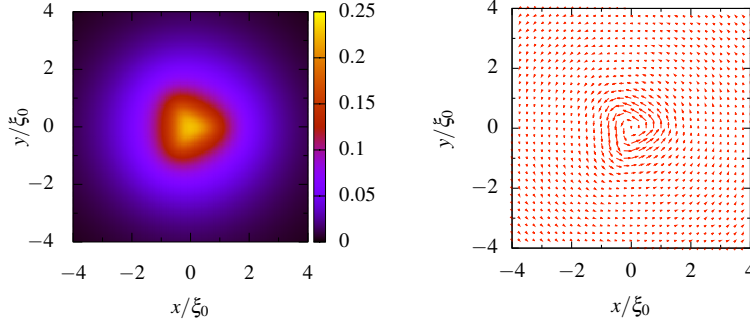
To solve (1), we used so-called Riccati-parametrization method[29,30] and solved the parametrized differential equation with a 4th- and 5th-order adaptive Runge-Kutta method. We used the cylindrical coordinate system and took 48 equally spaced points on the azimuthal coordinates. To improve the accuracy of numerical integration of the free-energy, we used the composite Gauss-Lobatto quadrature to choose discrete points  $r_i$  on the radial line. We divided the closed interval  $[0, 2.4]$  into 16 subintervals, applied the 7-points Gauss-Lobatto formula to each subinterval and obtained 97 discrete points  $x_i$ , and changed the variable as  $r_i = x_i + x_i^2/2 + x_i^3/3 + x_i^4/4$  in order to make the sampling points denser near the center and more sparse far from the vortex. We calculated the self-energy from the quasiclassical Green's functions via (4) and iterated the above until the self-energy sufficiently converged. After obtaining converged solution, we calculated the free-energy of vortices using (7). We changed initial profiles of the vortices so that the initial dominant- and induced-vortices were at separate positions, and repeated the same procedure as the above. Finally, we compared the resultant profiles and their free-energies to discuss stability.

### 3 Results and Discussion

As for the anti-parallel vortices, we only obtained circular axisymmetric vortices for all temperature that we studied ( $T/T_c = 0.1, 0.2, 0.3, 0.4$ , and  $0.5$ ), regardless of the initial profiles; in this case, the strong-coupling effect just modifies the shape of the vortex.

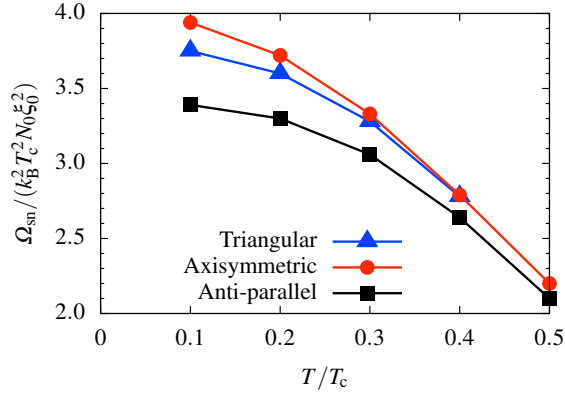


**Fig. 1** Profile of the off-diagonal part of the self-energy of the parallel vortex at  $T = 0.3T_c$  and  $\hbar\epsilon_n = \pi k_B T$ . Left-top: amplitude of dominant part  $(h(i\epsilon_n)(\tilde{\Sigma}_+)_{12}/T_c)$ , right-top: phase of dominant part  $(\arg(\tilde{\Sigma}_+)_{12})$ , left-down: amplitude of induced part  $(h(i\epsilon_n)(\tilde{\Sigma}_-)_{12}/T_c)$ , right-down: phase of induced part  $(\arg(\tilde{\Sigma}_-)_{12})$ . (Color figure online)



**Fig. 2** Electromagnetic quantities around the non-axisymmetric parallel vortex. Left: magnetic field, right: electric current density. (Color figure online)

On the other hand, at moderately low temperatures, a non-axisymmetric solution emerges for parallel vortices, when the initial vortex sufficiently breaks the axisymmetry. Figure 1 shows the non-axisymmetric profile of dominant and induced parts of off-diagonal self-energy at  $T/T_c = 0.3$  and  $\hbar\epsilon_n = \pi k_B T$ . There the vortex of dominant component forms triangle and those of the induced component split into three. Figure 2 shows the current density around the vortex. We can confirm that the electromagnetic quantities also break the axisymmetry.



**Fig. 3** Free energy of each vortex. Red-circle: circular parallel vortex, blue-triangle: triangular parallel vortex, black-square: circular anti-parallel vortex. (Color figure online)

When we calculated parallel vortices at  $T/T_c = 0.5$ , we found only an axisymmetric vortex: both circular and non-circular initial configurations of self-energy produce the same result. We thus conclude that unusual parallel vortices may not exist at high temperatures.

In Figure 3, we plot the free-energy of each vortex. We can see that at low temperatures, the non-axisymmetric vortex is more stable than symmetric one. We note that the symmetric anti-parallel vortex is more stable than the non-axisymmetric parallel vortex, at least under the parameters in this study.

There are many studies of non-axisymmetric vortices in spin triplet superfluids or superconductors with an isotropic Fermi surface. However, many of them have targeted vortices in the superfluid  $^3\text{He-B}$ [31,32,33,34,35,36,37], or an  $f$ -wave superconductor similar to the  $^3\text{He-B}$ [38]; these studies therefore cannot be compared with our work directly.

Tokuyasu, *et al.* have studied two-dimensional chiral  $p$ -wave superconductor within the GL theory in the weak- to strong-coupling regimes[41]. They have reported that non-axisymmetric vortices can emerge in some non-weak-coupling coefficients. However, the coefficients of the GL-functional of our target system fall into the same ones that we obtain in the weak-coupling limit (the  $\beta$  parameter in Ref. [41] is 0.5 in our system). Thus, the origin of non-axisymmetric vortices in our work is different from that of the previous work. This is also consistent with the fact that non-axisymmetric vortices only exist at low temperatures in the present work. Aoyama and Ikeda have reported that a vortex of  $^3\text{He-A}$  can be non-axisymmetric under the existence of anisotropic scatterers[39,40]. Their model is different from ours, and the relationship between their and our results considered an important but remaining issue.

## 4 Conclusion

In this study, we numerically found that a non-axisymmetric vortex metastably exists in strong-coupling chiral  $p$ -wave superconductors. This anomalous vortex is more stable than the axisymmetric parallel one at sufficiently low temperatures, but symmetric anti-parallel vortex is still most stable. The emergence of this anomalous vortex is a consequence of the strong-coupling effect because we did not obtain such a vortex with the conventional weak-coupling gap equation. To clarify the underlying energetics that makes the non-axisymmetric

vortex metastable is an interesting issue. The total phase diagram of this system is also left as a future issue.

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